

Lower Bounds on Nonnegative Signed Domination Parameters in Graphs

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Abstract

Let $1 \leq k \leq n$ be a positive integer. A *nonnegative signed k -subdominating function* is a function $f : V(G) \rightarrow \{-1, 1\}$ satisfying $\sum_{u \in N_G[v]} f(u) \geq 0$ for at least k vertices v of G . The value $\min \sum_{v \in V(G)} f(v)$, taking over all nonnegative signed k -subdominating functions f of G , is called the *nonnegative signed k -subdomination number* of G and denoted by $\gamma_{ks}^{NN}(G)$. When $k = |V(G)|$, $\gamma_{ks}^{NN}(G) = \gamma_s^{NN}(G)$ is the *nonnegative signed domination number*, introduced in [8]. In this paper, we investigate several sharp lower bounds of $\gamma_s^{NN}(G)$, which extend some presented lower bounds on $\gamma_s^{NN}(G)$. We also initiate the study of the nonnegative signed k -subdomination number in graphs and establish some sharp lower bounds for $\gamma_{ks}^{NN}(G)$ in terms of order and the degree sequence of a graph G .

Keywords: nonnegative signed domination number; nonnegative signed k -subdomination number

MSC 2000: 05C69

1 Introduction

Let G be a simple graph of order n with the vertex set $V(G)$ and size m with the edge set $E(G)$. We use [9] for terminology and notation, which are not defined here. The *minimum* and *maximum degrees* in graph G are denoted by $\delta(G)$ and $\Delta(G)$, respectively. A vertex $v \in V(G)$ is called an *odd (even)* vertex if $\deg_G(v)$ is odd (even). For a graph $G = (V, E)$, let V_o (V_e) be the set of odd (even) vertices with $n_o = |V_o|$ and $n_e = |V_e|$. If $X \subseteq V(G)$, then $G[X]$ is the *subgraph* of G induced by X . For disjoint subsets X and Y of vertices we let $E(X, Y)$ denote the set of edges between X and Y . The *open neighborhood* $N_G(v)$ of a vertex $v \in V(G)$ is the set of all vertices adjacent to v . Its *closed neighborhood* is $N_G[v] = N_G(v) \cup \{v\}$. In addition, the *open* and *closed neighborhoods* of a subset $S \subseteq V(G)$ are $N_G(S) = \cup_{v \in S} N_G(v)$ and $N_G[S] = N_G(S) \cup S$, respectively. The *degree* of a vertex $v \in V(G)$ is $\deg_G(v) = |N_G(v)|$. For a real-valued function $f : V(G) \rightarrow \mathbb{R}$ the weight of f is $\omega(f) = \sum_{v \in V(G)} f(v)$ and for a subset S of $V(G)$ we define $f(S) = \sum_{v \in S} f(v)$, so $\omega(f) = f(V(G))$. For a positive integer $1 \leq k \leq n$, a *signed k -subdominating function* (SkSDF) of G is a function $f : V(G) \rightarrow \{-1, 1\}$ such that $f(N_G[v]) \geq 1$ for

*Research supported by the research council of Faculty of Agriculture Engineering and Technology, University of Tehran, through grant No. 322870/1/03.

at least k vertices v of G . The *signed k -subdomination number* for a graph G is defined as $\gamma_{ks}(G) = \min\{\omega(f) \mid f \text{ is a SkSDF of } G\}$. The concept of the signed k -subdomination number was introduced and studied by Cockayne and Mynhardt [3]. A *nonnegative signed dominating function* (NNSDF) of G defined in [8] as a function $f : V(G) \rightarrow \{-1, 1\}$ such that $f(N_G[v]) \geq 0$ for all vertices v of G . The *nonnegative signed domination number* (NNSDN) of G is $\gamma_s^{NN}(G) = \min\{\omega(f) \mid f \text{ is an NNSDF of } G\}$.

We now introduce a *nonnegative signed k -subdominating function* (NNSkSDF) of G for a positive integer $1 \leq k \leq n$ as a function $f : V(G) \rightarrow \{-1, 1\}$ such that $f(N_G[v]) \geq 0$ for at least k vertices v of G . We define the *nonnegative signed k -subdomination number* (NNSkSDN) of G by $\gamma_{ks}^{NN}(G) = \min\{\omega(f) \mid f \text{ is a NNSkSDF of } G\}$. A nonnegative signed k -subdominating function of weight $\gamma_{ks}^{NN}(G)$ is called a $\gamma_{ks}^{NN}(G)$ -function. Note that $\gamma_{ns}^{NN}(G) = \gamma_s^{NN}(G)$. Since every signed k -subdominating function of G is a nonnegative signed k -subdominating function, we deduce that

$$\gamma_{ks}^{NN}(G) \leq \gamma_{ks}(G).$$

For a function $f : V(G) \rightarrow \{-1, 1\}$ of G we define $P = \{v \in V(G) \mid f(v) = 1\}$, $M = \{v \in V(G) \mid f(v) = -1\}$, and $C_f = \{v \in V(G) \mid f(N_G[v]) \geq 0\}$.

In this paper, we establish some new lower bounds on $\gamma_s^{NN}(G)$ for a general graph in terms of various different graph parameters. Some of these bounds improve several lower bounds on $\gamma_s^{NN}(G)$ presented in [1, 8]. We also initiate the study of nonnegative signed k -subdomination numbers in graphs, and present some sharp lower bounds for $\gamma_{ks}^{NN}(G)$ in terms of the order and the degree sequence of a graph G .

Observation 1. Let f be an NNSkSDF of G . For $v \in C_f$ if v is an even vertex, then $f(N_G[v]) \geq 1$ while $f(N_G[v]) \geq 0$ if v is an odd vertex.

Observation 2. Let $1 \leq k \leq n$ be a positive integer. For any even graph G ,

$$\gamma_{ks}^{NN}(G) = \gamma_{ks}(G).$$

In this paper, we make use of the following results.

Theorem A. [1] Let G be a graph of order n and size m . Then

$$\gamma_s^{NN}(G) \geq \frac{n}{2} - m.$$

Theorem B. [1] Let G be a graph of order n , size m and minimum degree δ . Then

$$\gamma_s^{NN}(G) \geq \frac{-4m + 3n \lceil \frac{\delta+1}{2} \rceil - n}{3 \lceil \frac{\delta+1}{2} \rceil + 1}.$$

Theorem C. [4] For $n \geq 3$,

$$\gamma_s(C_n) = \begin{cases} n/3 & \text{if } n \equiv 0 \pmod{3}, \\ \lfloor \frac{n}{3} \rfloor + 1 & \text{if } n \equiv 1 \pmod{3}, \\ \lfloor \frac{n}{3} \rfloor + 2 & \text{if } n \equiv 2 \pmod{3}. \end{cases}$$

Corollary 3. [7] For any r -regular graph G of order n , $\gamma_s(G) \geq \frac{n}{r+1}$, for r even. Furthermore this bound is sharp.

Theorem D. [8] Let K_n be a complete graph. Then $\gamma_s^{NN}(K_n) = 0$ when n is even and $\gamma_s^{NN}(K_n) = 1$ when n is odd.

Theorem E. [8] For any graph G with maximum degree Δ and minimum degree δ , we have

$$\gamma_s^{NN}(G) \geq \frac{\delta - \Delta}{\delta + \Delta + 2}n.$$

2 Lower bounds on the NNSDNs of graphs

In this section, we present some new sharp lower bounds for $\gamma_s^{NN}(G)$ by using n_e as the number of even vertices in a graph G . We begin with the following lemma.

Lemma 4. Let f be an NNSDF of a simple connected graph G . Then,

1. $\sum_{v \in P} \deg_G(v) \geq n + n_e - 2 |P| + \sum_{v \in M} \deg_G(v)$.
2. $\sum_{v \in P} \deg_{G[P]}(v) \geq \sum_{v \in P} \lceil \frac{\deg_G(v)-1}{2} \rceil$.

Proof. For $v \in V(G)$, let $\deg_P(v)$ and $\deg_M(v)$ denote the numbers of vertices of P and M , respectively, which are adjacent to v . Clearly, $\deg_G(v) = \deg_M(v) + \deg_P(v)$. Since $f(N_G[v]) \geq 0$, for every $v \in P$, $\deg_M(v) \leq \deg_P(v) + 1$, and for every $v \in M$, $\deg_P(v) \geq \deg_M(v) + 1$. Hence, if $v \in P$, then $\deg_M(v) \leq \lfloor \frac{\deg_G(v)+1}{2} \rfloor$ and if $v \in M$, then $\deg_P(v) \geq \lceil \frac{\deg_G(v)+1}{2} \rceil$.

1. Counting the number of edges in $E(P, M)$ in two ways, we can deduce that

$$\sum_{v \in M} \lceil \frac{\deg_G(v)+1}{2} \rceil \leq |E(P, M)| \leq \sum_{v \in P} \lfloor \frac{\deg_G(v)+1}{2} \rfloor$$

It follows that

$$\begin{aligned} \sum_{v \in P} \deg_G(v) + |P| &\geq \sum_{v \in V} \lceil \frac{\deg_G(v)+1}{2} \rceil \\ &= \sum_{v \in V_o} \frac{\deg_G(v)+1}{2} + \sum_{v \in V_e} \frac{\deg_G(v)+2}{2} \\ &= \sum_{v \in V} \frac{\deg_G(v)}{2} + n_e + \frac{n_o}{2} \\ &= \sum_{v \in P} \frac{\deg_G(v)}{2} + \sum_{v \in M} \frac{\deg_G(v)}{2} + n_e + \frac{n_o}{2}, \end{aligned}$$

which implies that

$$\sum_{v \in P} \frac{\deg_G(v)}{2} \geq \sum_{v \in M} \frac{\deg_G(v)}{2} + n_e + \frac{n_o}{2} - |P|.$$

Hence,

$$\sum_{v \in P} \deg_G(v) \geq \sum_{v \in M} \deg_G(v) + n + n_e - 2 |P|.$$

2. Consider the subgraph $G[P]$ induced by P . We have $\deg_{G[P]}(v) = \deg_P(v)$ for each $v \in P$. Since $\deg_P(v) \geq \lceil \frac{\deg_G(v)-1}{2} \rceil$ for each $v \in P$, we have

$$\sum_{v \in P} \deg_{G[P]}(v) \geq \sum_{v \in P} \lceil \frac{\deg_G(v)-1}{2} \rceil.$$

□

In the next theorem we present some lower bounds on $\gamma_s^{NN}(G)$. By using Lemma 4 and graph parameters such as order, size, number of even vertices, maximum and minimum degrees we obtain some new lower bounds for $\gamma_s^{NN}(G)$. These new results are independent from each other.

Theorem 5. Let G be a simple connected graph of order n , minimum degree δ , maximum degree Δ and the number of even vertices n_e . Then

1. $\gamma_s^{NN}(G) \geq \frac{n\delta - n\Delta + 2n_e}{\Delta + \delta + 2},$
2. $\gamma_s^{NN}(G) \geq \frac{2m + n_e - n\Delta}{\Delta + 1},$
3. $\gamma_s^{NN}(G) \geq \frac{n\delta + n_e - 2m}{\delta + 1},$
4. $\gamma_s^{NN}(G) \geq \lceil \frac{-(\delta + 1) + \sqrt{(\delta + 1)^2 + 8(n\delta + n + n_e)}}{2} - n \rceil,$
5. $\gamma_s^{NN}(G) \geq \lceil \sqrt{2m + n + n_e} - n \rceil.$

Proof. According to Lemma 4, we have

$$\sum_{v \in M} \deg_G(v) + n + n_e - 2 \mid P \mid \leq \sum_{v \in P} \deg_G(v). \quad (1)$$

1. Since $\delta \leq \deg_G(v) \leq \Delta$ for each $v \in V(G)$, inequality (1) follows that

$$\delta n - \mid P \mid \delta + n + n_e - 2 \mid P \mid \leq \sum_{v \in P} \deg_G(v) \leq \Delta \mid P \mid.$$

From this inequality, it is deduced that

$$\mid P \mid \geq \frac{\delta n + n + n_e}{\Delta + \delta + 2}.$$

Hence,

$$\gamma_s^{NN}(G) = 2 \mid P \mid - n \geq \frac{n\delta - n\Delta + 2n_e}{\Delta + \delta + 2}.$$

2. Since $2m = \sum_{v \in V} \deg_G(v)$ and $\deg_G(v) \leq \Delta$ for each $v \in V(G)$, by inequality (1) it follows that

$$\begin{aligned} 2m + n + n_e - 2 \mid P \mid &= \sum_{v \in V} \deg_G(v) + n + n_e - 2 \mid P \mid \\ &\leq 2 \sum_{v \in P} \deg_G(v) \\ &\leq 2\Delta \mid P \mid. \end{aligned}$$

Therefore, $2 \mid P \mid \geq \frac{2m + n + n_e}{\Delta + 1}$, and $\gamma_s^{NN}(G) \geq \frac{2m + n_e - n\Delta}{\Delta + 1}$, as desired.

3. Using inequality (1) and the facts $2m = \sum_{v \in V} \deg_G(v)$ and $\deg_G(v) \geq \delta$ for any $v \in V(G)$, we have

$$\begin{aligned} 2m &= \sum_{v \in V} \deg_G(v) \\ &\geq 2 \sum_{v \in M} \deg_G(v) + n + n_e - 2 |P| \\ &\geq 2n\delta - 2\delta |P| + n + n_e - 2 |P| \end{aligned}$$

It follows that

$$2 |P| \geq \frac{2n\delta + n + n_e - 2m}{\delta + 1}.$$

Thus, $\gamma_s^{NN}(G) \geq \frac{n\delta + n_e - 2m}{\delta + 1}$, as desired.

4. Consider $G[P]$. According to Lemma 4, we have

$$\sum_{v \in P} \deg_{G[P]}(v) \geq \sum_{v \in P} \lceil \frac{\deg_G(v) - 1}{2} \rceil \geq \sum_{v \in P} \frac{\deg_G(v) - 1}{2}.$$

On the other hand, since $G[P]$ is a simple connected graph,

$$\sum_{v \in P} \deg_{G[P]}(v) \leq |P| (|P| - 1).$$

Thus,

$$\begin{aligned} 2 |P| (|P| - 1) &\geq \sum_{v \in P} \deg_G(v) - |P| \\ &\geq \sum_{v \in M} \deg_G(v) + n + n_e - 3 |P| \\ &\geq n\delta - \delta |P| + n + n_e - 3 |P|. \end{aligned}$$

This implies that

$$2 |P|^2 + (\delta + 1) |P| - (n\delta + n + n_e) \geq 0.$$

Therefore,

$$2 |P| \geq \frac{-(\delta + 1) + \sqrt{(\delta + 1)^2 + 8(n\delta + n + n_e)}}{2},$$

and hence $\gamma_s^{NN}(G) \geq \lceil \frac{-(\delta + 1) + \sqrt{(\delta + 1)^2 + 8(n\delta + n + n_e)}}{2} - n \rceil$, as desired.

5. By Parts (4) and (2) we have

$$2 \sum_{v \in P} \deg_G(v) \leq 4 |P|^2 - 2 |P|,$$

and

$$2 \sum_{v \in P} \deg_G(v) \geq 2m + n + n_e - 2 |P|,$$

respectively. So,

$$4 \mid P \mid^2 \geq 2m + n + n_e,$$

which implies that

$$\mid P \mid \geq \frac{\sqrt{2m + n + n_e}}{2}.$$

Thus, $\gamma_s^{NN}(G) \geq \lceil \sqrt{2m + n + n_e} - n \rceil$, as desired.

□

From Theorem 5 (1)–(3), we have the following result. For $k = n$ by Observation 2 when r is even, we have the same bound presented in Corollary 3 by Henning.

Corollary 6. For $r \geq 1$, if G is an r -regular graph of order n , then

$$\gamma_s^{NN}(G) \geq \begin{cases} \frac{n}{r+1} & \text{if } r \text{ is even,} \\ 0 & \text{if } r \text{ is odd.} \end{cases}$$

Furthermore, these bounds are sharp.

In order to show that the bounds presented in Theorem 5 are sharp, we will give a graph G and construct a $\gamma_s^{NN}(G)$ -function f such that $w(f)$ achieves the lower bounds, and thus the lower bounds are sharp. We also illuminate that our bounds for some of these graphs are attainable while the corresponding bounds given in Theorems A, B, and E are not. In fact, a trivial examples such $G \in \{K_n, C_n\}$ is sufficient for this. It is easy to see that $\gamma_s^{NN}(K_n)$ obtains all the five bounds in Theorem 5 while the bound in Theorem A shows that $\gamma_s^{NN}(K_n) \geq \frac{2n-n^2}{2}$ and the bound in Theorem B is not more than $\frac{5n-n^2}{3n+5}$. As an other example, $\gamma_s^{NN}(C_n)$ attains the lower bounds in Theorem 5 (1)–(3), when $n \equiv 0 \pmod{3}$ while the bounds in Theorems A, B and E are not more than $\frac{n}{7}$. Besides, we can construct a non-complete graph with an arbitrary large order whose reaches the lower bounds in Theorem 5 (1)–(3) as follows. Letting t be a positive integer, we consider a cycle of length $2t$. For every edge, we include an additional vertex being adjacent to both endpoints of the corresponding edge. The obtained graph is denoted by G . It is easy to check that the graph G is a graph with $n = 4t$, $m = 6t$, $\delta = 2$, $\Delta = 4$ and $n_e = 4t$. Define a function $f : V(G) \rightarrow \{-1, 1\}$ as follows: $f(v) = 1$ for $v \in V(C_{2t})$ and $f(v) = -1$ for $v \in V(G) \setminus V(C_{2t})$. It is easy to verify that f is a $\gamma_s^{NN}(G)$ -function with $w(f) = 0$ and bounds in Theorem 5 (1)–(3) are also 0, which implies that G is sharp for these bounds. However, $\gamma_s^{NN}(G)$ does not attain the corresponding bounds given in Theorems A, B and E, which are $-4t$, $\lceil -\frac{4t}{7} \rceil$, and $-t$, respectively. Next, we show that there is also a graph G different from K_n such that $\gamma_s^{NN}(G)$ reaches lower bounds in Theorem 5 (4)–(5). Let H be the Hajós graph. We can verify that $\gamma_s^{NN}(H) = 0$, and H is sharp for presented bounds in Theorem 5 (4)–(5).

3 Lower bounds on the NNSkSDNs of graphs

In this section, we initiate the study of the nonnegative signed k -subdomination number in graphs. we present some lower bounds on the nonnegative signed k -subdomination number of a graph in terms of the order and the degree sequence. We begin with the following lemma.

Lemma 7. Let G be a graph and $1 \leq k \leq n$ be a positive integer. Let f be a $\gamma_{ks}^{NN}(G)$ -function. Let $P_1 = P \cap C_f$, $P_2 = P \setminus P_1$, $M_1 = M \cap C_f$ and $M_2 = M \setminus M_1$. Then,

$$\sum_{v \in P} \deg_G(v) + |P_1| \geq \sum_{v \in P_1 \cup M_1} \lceil \frac{\deg_G(v) + 1}{2} \rceil.$$

Proof. Note that if $v \in V(G)$, then $\deg_G(v) = \deg_P(v) + \deg_M(v)$. For $v \in P_1 \cup M_1$, $f(N_G[v]) \geq 0$. So, if $v \in P_1$, then $\deg_P(v) \geq \lceil \frac{\deg_G(v) - 1}{2} \rceil$ and $\deg_M(v) \leq \lfloor \frac{\deg_G(v) + 1}{2} \rfloor$. Similarly, if $v \in M_1$, then $\deg_P(v) \geq \lceil \frac{\deg_G(v) + 1}{2} \rceil$ and $\deg_M(v) \leq \lfloor \frac{\deg_G(v) - 1}{2} \rfloor$.

Counting the number of edges in $E(P, M)$ in two ways, we conclude that

$$\sum_{v \in M_1} \lceil \frac{\deg_G(v) + 1}{2} \rceil + \sum_{v \in M_2} \deg_P(v) \leq \sum_{v \in P_1} \lfloor \frac{\deg_G(v) + 1}{2} \rfloor + \sum_{v \in P_2} \deg_M(v).$$

By adding $\sum_{v \in P_1} \lceil \frac{\deg_G(v) + 1}{2} \rceil$ to the both sides of the inequality we have

$$\begin{aligned} \sum_{v \in P} \deg_G(v) + |P_1| &\geq \sum_{v \in P_1 \cup M_1} \lceil \frac{\deg_G(v) + 1}{2} \rceil + \sum_{v \in M_2} \deg_G(v) \\ &\geq \sum_{v \in P_1 \cup M_1} \lceil \frac{\deg_G(v) + 1}{2} \rceil, \end{aligned}$$

and this completes the proof. \square

Theorem 8. For any graph G with degree sequence $d_1 \leq d_2 \leq \dots \leq d_n$ and any positive integer $1 \leq k \leq n$,

1. $\gamma_{ks}^{NN}(G) \geq \frac{2 \sum_{i=1}^k \lceil \frac{d_i + 1}{2} \rceil}{\Delta + 1} - n.$
2. $\gamma_{ks}^{NN}(G) \geq \frac{n\delta - 4m - n + 2 \sum_{i=1}^k \lceil \frac{d_i + 1}{2} \rceil}{\delta + 1}.$

Furthermore, these bounds are sharp.

Proof. Considering Lemma 7 we have

$$\sum_{v \in P} \deg_G(v) + |P_1| \geq \sum_{v \in P_1 \cup M_1} \lceil \frac{\deg_G(v) + 1}{2} \rceil. \quad (2)$$

1. Since $\delta \leq \deg_G(v) \leq \Delta$ for each $v \in V(G)$, inequality (2) follows that

$$\Delta |P| + |P| \geq \sum_{v \in P} \deg_G(v) + |P_1| \geq \sum_{v \in P_1 \cup M_1} \lceil \frac{\deg_G(v) + 1}{2} \rceil.$$

Note that $P_1 \cap M_1 = \emptyset$ and $|P_1 \cup M_1| = |P_1| + |M_1| \geq k$. So,

$$|P| \geq \frac{\sum_{v \in P_1 \cup M_1} \lceil \frac{\deg_G(v) + 1}{2} \rceil}{\Delta + 1} \geq \frac{\sum_{i=1}^k \lceil \frac{d_i + 1}{2} \rceil}{\Delta + 1}.$$

Thus,

$$\gamma_{ks}^{NN}(G) = 2|P| - n \geq \frac{2 \sum_{i=1}^k \lceil \frac{d_i + 1}{2} \rceil}{\Delta + 1} - n.$$

2. Obviously, $2m = \sum_{v \in V(G)} \deg_G(v) = \sum_{v \in P} \deg_G(v) + \sum_{v \in M} \deg_G(v)$. If we add $|P_1|$ to the both sides of this equality, then by Lemma 7 we deduce that

$$\begin{aligned} |P| \geq |P_1| &\geq -2m + \sum_{v \in P_1 \cup M_1} \lceil \frac{\deg_G(v) + 1}{2} \rceil + \sum_{v \in M} \deg_G(v) \\ &\geq -2m + \sum_{i=1}^k \lceil \frac{d_i + 1}{2} \rceil + \delta n - \delta |P|. \end{aligned}$$

Therefore,

$$|P| \geq \frac{n\delta - 2m + \sum_{i=1}^k \lceil \frac{d_i + 1}{2} \rceil}{\delta + 1},$$

and hence,

$$\gamma_{ks}^{NN}(G) = 2|P| - n \geq \frac{n\delta - 4m - n + 2 \sum_{i=1}^k \lceil \frac{d_i + 1}{2} \rceil}{\delta + 1}.$$

Now suppose that $k = n$, considering that $2 \sum_{i=1}^n \lceil \frac{d_i + 1}{2} \rceil = 2m + n + n_e$, we can immediately obtain those two bounds in Theorem 5 (2) and (3) from the lower bounds of Theorem 8, respectively. Since the bounds in Theorem 5 are sharp, so there exist graphs whose $\gamma_{ks}^{NN}(G)$ receive the bounds in Theorem 8. Therefore, these bounds are sharp. \square

As an immediate consequence of Theorem 8 we have the following result.

Corollary 9. [5] For every r -regular graph G of order n , $\gamma_{ks}(G) \geq \frac{k(r+2)}{r+1} - n$ for r even.

Corollary 10. For $r \geq 1$, if G is an r -regular graph of order n , then

$$\gamma_{ks}^{NN}(G) \geq \begin{cases} \frac{k(r+2)}{r+1} - n & \text{if } r \text{ is even,} \\ k - n & \text{if } r \text{ is odd.} \end{cases}$$

Furthermore, these bounds are sharp.

Clearly, if r is even, then by Observation 2 we have the same given bound in Corollary ??.

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